

STABILITY CONSTANTS OF THE WEAK* FIXED POINT PROPERTY FOR THE SPACE ℓ_1

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ABSTRACT. The main aim of the paper is to study some quantitative aspects of the stability of the weak* fixed point property for nonexpansive maps in ℓ_1 (shortly, w^* -fpp). We focus on two complementary approaches to this topic. First, given a predual X of ℓ_1 such that the $\sigma(\ell_1, X)$ -fpp holds, we precisely establish how far, with respect to the Banach-Mazur distance, we can move from X without losing the w^* -fpp. The interesting point to note here is that our estimate depends only on the smallest radius of the ball in ℓ_1 containing all $\sigma(\ell_1, X)$ -cluster points of the extreme points of the unit ball. Second, we pass to consider the stability of the w^* -fpp in the restricted framework of preduals of ℓ_1 . Namely, we show that every predual X of ℓ_1 with a distance from c_0 strictly less than 3, induces a weak* topology on ℓ_1 such that the $\sigma(\ell_1, X)$ -fpp holds.

1. INTRODUCTION

Let X be an infinite dimensional real Banach space and let us denote by B_X its closed unit ball and by S_X its unit sphere. A nonempty bounded closed and convex subset C of X has the fixed point property (shortly, fpp) if each nonexpansive mapping (i.e., the mapping $T : C \rightarrow C$ such that $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$) has a fixed point. A dual space X^* is said to have the $\sigma(X^*, X)$ -fixed point property ($\sigma(X^*, X)$ -fpp or, shortly, w^* -fpp when no confusion can arise) if every nonempty, convex, $\sigma(X^*, X)$ -compact subset C of X^* has the fpp.

The study of the $\sigma(X^*, X)$ -fpp reveals to be of special interest whenever a dual space has different preduals. For instance, this situation occurs when we consider the space ℓ_1 and its preduals c_0 and c where it is well-known (see [12]) that ℓ_1 has the $\sigma(\ell_1, c_0)$ -fpp whereas it lacks the $\sigma(\ell_1, c)$ -fpp. Necessary and sufficient conditions for a predual X of ℓ_1 to be a space such that the $\sigma(\ell_1, X)$ -fpp holds are proved in [5]. The present paper concerns the investigation of the stability of the $\sigma(\ell_1, X)$ -fpp carrying on the study developed in [7]. We recall that stability of fixed point property deals with the following question: let us suppose that a Banach space X has the fixed point property and Y is a Banach space isomorphic to X with "small" Banach-Mazur distance, does Y have fixed point property? This problem has been widely studied for fpp and only occasionally for weak* topology (see [8, 16]). In [7], we established a characterization of stability of the $\sigma(\ell_1, X)$ -fpp by means of a geometrical property. In the present paper we wish to investigate some quantitative aspects of stability of the $\sigma(\ell_1, X)$ -fpp. To this aim, when X^* enjoys the $\sigma(X^*, X)$ -fpp, it reveals to be useful to introduce the stability constant:

$$\gamma^*(X) = \sup \{ \gamma \geq 1 : Y^* \text{ has the } \sigma(Y^*, Y)\text{-fpp whenever } d(X, Y) \leq \gamma \},$$

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where $d(X, Y)$ denotes the Banach-Mazur distance between X and Y . Under the assumption that X is a predual of ℓ_1 , Section 2 establishes the relation between the stability constant and the constant

$$r^*(X) = \inf \{r > 0 : (\text{ext}(B_{\ell_1}))' \subset rB_{\ell_1}\}$$

where $(\text{ext}(B_{\ell_1}))'$ denotes the set of $\sigma(\ell_1, X)$ -limit points of the extreme points of B_{ℓ_1} . It is worth pointing out that the spaces satisfying $(\text{ext}(B_{X^*}))' \subset rB_{X^*}$ with $r < 1$ have already been studied in the framework of polyhedrality for Banach spaces (see [6, 7, 9, 10]). Moreover, it is known that if $r^*(X) = 0$, then $X = c_0$ (see [9]) and in this case $\gamma^*(c_0) \leq 2$ by a result by Lim [13]. A lower bound for $\gamma^*(X)$ can be obtained from the key result of [7]. Indeed, in that paper we proved that if $r^*(X) \in [0, 1]$, then

$$\gamma^*(X) \geq \frac{2}{1 + r^*(X)}.$$

Moreover, from Theorem 3.4 in [7] we know that if $r^*(X) = 1$, then $\gamma^*(X) = 1$. The main result of Section 2 shows that for $r^*(X) \in (0, 1)$ also the reverse inequality holds true. Therefore, we obtain that

$$\gamma^*(X) = \frac{2}{1 + r^*(X)}.$$

The stability constant $\gamma^*(X)$ takes into account a broad family of perturbations of the space X . Indeed, if Y is isomorphic to X , then both the classes of $\sigma(Y^*, Y)$ -compact convex sets and of nonexpansive mappings on Y^* are different from those in X^* . It may be interesting to deal with the stability of the $\sigma(\ell_1, X)$ -fpp in the restricted framework of the preduals of ℓ_1 , hence by allowing only modifications to the class of $\sigma(Y^*, Y)$ -compact convex sets. Namely, in Section 3, we deal with the estimation of the following constant:

$$\eta^*(X) = \sup \{ \eta \geq 1 : Y^* = \ell_1, d(X, Y) \leq \eta \Rightarrow Y^* \text{ has the } \sigma(\ell_1, Y)\text{-fpp} \}.$$

where X is a predual of ℓ_1 that enjoys the $\sigma(\ell_1, X)$ -fpp. The main result of this section gives a sharp estimation of $\eta^*(c_0)$. Indeed, we prove that $\eta^*(c_0) = 3$. The estimate we obtain is interesting since it shows that we should move as far as c is from c_0 in order to lack the w^* -fpp. Finally, we provide an example of a predual X of ℓ_1 such that the $\sigma(\ell_1, X)$ -fpp holds, whereas the Banach-Mazur distance between c_0 and X is equal to 3.

2. A QUANTITATIVE VIEW ON STABILITY OF WEAK* FIXED POINT PROPERTY IN ℓ_1

This section is devoted to study the stability of the $\sigma(\ell_1, X)$ -fpp. In particular we provide a sharp estimation of the constant $\gamma^*(X)$ as defined in the introduction. In order to prove the main result of this section we need some preliminary steps. We begin with the following simple lemma. Probably it is well known, but we were not able to find a suitable reference.

Lemma 2.1. *Let $x_n^* \subset X^*$ be a sequence norm convergent to x^* . Then*

$$\lim_{n \rightarrow \infty} d(\ker x^*, \ker x_n^*) = 1.$$

Proof. Let us consider a projection $P : X \rightarrow \ker x^*$. It is well known that $P(x) = x - x^*(x)z$, where $x^*(z) = 1$. We can assume that $x_n^*(z) > 0$. Then, let $\lambda_n \in \mathbb{R}$ be such that $x_n^*(\lambda_n z) = 1$ for every n . It holds

$$|1 - \lambda_n| = |x_n^*(\lambda_n z) - x^*(\lambda_n z)| \leq |\lambda_n| \|z\| \|x^* - x_n^*\|.$$

Hence, $\lim_{n \rightarrow \infty} \lambda_n = 1$. Now, we define a sequence of projections $P_n : X \rightarrow \ker x_n^*$ such that $P_n(x) = x - \lambda_n x_n^*(x)z$. Since

$$\|P(x) - P_n(x)\| \leq \|\lambda_n x_n^* - x^*\| \|x\| \|z\|,$$

we have $\lim_{n \rightarrow \infty} \|P - P_n\| = 0$. Now we recall the definition of operator opening between two closed subspaces Y and Z of the space X (see [15]). Let us denote by

$$r_0(Y, Z) = \inf \{ \|\psi - I\| : \psi(Y) = Z, \psi \text{ invertible linear operator on } X \},$$

then the operator opening between Y and Z is defined by $r(Y, Z) = \max \{r_0(Y, Z), r_0(Z, Y)\}$. Theorem 4.2 (e) in [15] implies

$$\lim_{n \rightarrow \infty} r(\ker x^*, \ker x_n^*) = 0.$$

Finally, Proposition 6.1 in [15] allows us to conclude that

$$\lim_{n \rightarrow \infty} d(\ker x^*, \ker x_n^*) = 1.$$

□

Now, we pass to consider the properties of a suitable renorming of some hyperplanes of c , the space of convergent sequences. We recall that the standard duality between c and ℓ_1 is defined by

$$x^*(x) = x^*(1) \lim_i x(i) + \sum_{j=1}^{\infty} x^*(j+1)x(j)$$

for every $x = (x(1), x(2), \dots) \in c$ and $x^* = (x^*(1), x^*(2), \dots) \in \ell_1$.

For every $\alpha = (\alpha(1), \alpha(2), \dots) \in B_{\ell_1}$, we define the following hyperplane of c :

$$W_\alpha = \left\{ x = (x(1), x(2), \dots) \in c : \lim_{i \rightarrow \infty} x(i) = \sum_{i=1}^{+\infty} \alpha(i)x(i) \right\}.$$

For a detailed study of this class of spaces we refer the reader to [4] and [5]. Here we recall only that $W_\alpha^* = \ell_1$.

Here and subsequently we adopt the following notations: $\|\cdot\|_\infty$ and $|\cdot|_{\ell_1}$ denote respectively the standard norm in c and ℓ_1 . Let $n \in \mathbb{N}$. For each sequence $x = (x(1), x(2), \dots)$ we put $P_n(x) = (x(1), \dots, x(n), 0, 0, \dots)$ and $R_n x = x - P_n x$, and x^+ and x^- denotes the positive and negative part of x , respectively.

Propositions 2.2 and 2.3 presented below form one of the main steps in the proof of the main result of this section.

Proposition 2.2. *Let $e^* = (e^*(1), \dots, e^*(n), 0, 0, \dots) \in \ell_1$ with $r_n := |e^*|_{\ell_1} \in (0, 1)$. For all $x \in W_{e^*}$, define*

$$\|x\|_n = (\|R_n x^+\|_\infty \vee r_n \|R_n x^-\|_\infty + \|R_n x^-\|_\infty \vee r_n \|R_n x^+\|_\infty) \vee (1+r_n) \|P_n x\|_\infty.$$

Then

$$(W_{e^*}, \|\cdot\|_n)^* = (\ell_1, |\cdot|_n),$$

where

$$|f|_n = \max \left\{ \frac{r_n |R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1}}{1+r_n}, \frac{|R_n f^+|_{\ell_1} + r_n |R_n f^-|_{\ell_1}}{1+r_n} \right\} + \frac{|P_n f|_{\ell_1}}{1+r_n},$$

and a duality map $\phi : \ell_1 \rightarrow W_{e^*}$ is defined by

$$(\phi(f))(x) = \sum_{j=1}^{+\infty} x(j)f(j),$$

where $f = (f(1), f(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in W_{e^*}$.

Proof. First, we begin by noticing that $\|\cdot\|_n$ is a norm equivalent to the $\|\cdot\|_\infty$ norm,

$$(1+r_n)\|x\|_\infty = (1+r_n)(\|R_n x\|_\infty \vee \|P_n x\|_\infty) \leq \|x\|_n \leq 2(\|R_n x\|_\infty \vee \|P_n x\|_\infty) = 2\|x\|_\infty$$

so, from Theorem 4.3 in [4], we know that the dual of $(W_{e^*}, \|\cdot\|_n)$ is representable by ℓ_1 with duality map $\phi : \ell_1 \rightarrow W_{e^*}$ defined by

$$(\phi(f))(x) = \sum_{j=1}^{+\infty} x(j)f(j),$$

where $f = (f(1), f(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in W_{e^*}$. Therefore it suffices to show that

$$|f|_n = \sup \left\{ \sum_{i=1}^{\infty} x(i)f(i) : x \in W_{e^*}, \|x\|_n \leq 1 \right\}$$

for each $f \in \ell_1$. As in [13], the supremum can be taken again over x satisfying $x(i)f(i) \geq 0$. In case $x(i)f(i) < 0$, replace it by 0 when estimating from above. Also, notice that

$$\frac{1}{2}|f|_{\ell_1} = \frac{|R_n f|_{\ell_1} + |P_n f|_{\ell_1}}{2} \leq |f|_n \leq \frac{|R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1} + |P_n f|_{\ell_1}}{1+r_n} = \frac{1}{1+r_n}|f|_{\ell_1}.$$

Without loss of generality, one can assume that

$$|f|_n = \frac{r_n}{1+r_n}|R_n f^+|_{\ell_1} + \frac{1}{1+r_n}|R_n f^-|_{\ell_1} + \frac{1}{1+r_n}|P_n f|_{\ell_1},$$

and so

$$|R_n f^-|_{\ell_1} \geq |R_n f^+|_{\ell_1} \quad (\heartsuit).$$

There are three cases to consider.

Case 1. Assume that

$$(2.1) \quad \|R_n x^+\|_\infty \vee r_n \|R_n x^-\|_\infty = |x^+(i)|$$

and

$$(2.2) \quad \|R_n x^-\|_\infty \vee r_n \|R_n x^+\|_\infty = |x^-(j)|$$

for some $i, j \geq n+1$. Then

$$\|x\|_n = (|x^+(i)| + |x^-(j)|) \vee (1+r_n)\|P_n x\|_\infty.$$

SubCase 1.1. If $(1+r_n)\|P_n x\|_\infty \leq |x^+(i)| + |x^-(j)|$, then

$$\begin{aligned} (1+r_n)f(x) &= (1+r_n) \sum_{k=1}^n x(k)f(k) + (1+r_n) \sum_{k=n+1}^{\infty} x(k)f(k) \\ &\leq (|x^+(i)| + |x^-(j)|) |P_n f|_{\ell_1} + (1+r_n) |x^+(i)| |R_n f^+|_{\ell_1} \\ &\quad + (1+r_n) |x^-(j)| |R_n f^-|_{\ell_1} \\ &\leq (|x^+(i)| + |x^-(j)|) \left(|P_n f|_{\ell_1} + r_n |R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1} \right) \end{aligned}$$

and the last inequality holds since $r_n |x^-(j)| \leq |x^+(i)|$ by (2.1) and $|R_n f^+|_{\ell_1} \leq |R_n f^-|_{\ell_1}$ by (\heartsuit) . Thus, $f(x) \leq \|x\|_n |f|_n$.

SubCase 1.2. If $|x^+(i)| + |x^-(j)| \leq (1+r_n)\|P_n x\|_\infty$, then from (2.1) we obtain $r_n |x^-(j)| \leq |x^+(i)|$ and so $(1+r_n)|x^-(j)| \leq (1+r_n)\|P_n x\|_\infty$, or equivalently $|x^-(j)| \leq \|P_n x\|_\infty$. Now we have

$$\begin{aligned}
f(x) &\leq \sum_{k=1}^n |x(k)| |f(k)| + |x^+(i)| |R_n f^+|_{\ell_1} + |x^-(j)| |R_n f^-|_{\ell_1} \\
&\leq \|P_n x\|_\infty \left(|P_n f|_{\ell_1} + r_n |R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1} \right).
\end{aligned}$$

This time the last inequality holds since $|x^+(i)| - r_n \|P_n x\|_\infty \leq \|P_n x\|_\infty - |x^-(j)|$, $0 \leq \|P_n x\|_\infty - |x^-(j)|$ and $|R_n f^+|_{\ell_1} \leq |R_n f^-|_{\ell_1}$ by (\heartsuit) . Therefore, we obtain again that $f(x) \leq \|x\|_n |f|_n$.

Case 2. $\|R_n x^+\|_\infty \vee r_n \|R_n x^-\|_\infty = |x^+(i)|$, $\|R_n x^-\|_\infty \vee r_n \|R_n x^+\|_\infty = r_n |x^+(i)|$ for some $i \geq n+1$ and so $\|x\|_n = (1+r_n) |x^+(i)| \vee (1+r_n) \|P_n x\|_\infty = (1+r_n) \|x\|_\infty$. This further implies $\|R_n x^-\|_\infty \leq r_n \|x\|_\infty$ and so

$$\begin{aligned}
f(x) &\leq \|x\|_\infty |P_n f|_{\ell_1} + \|x\|_\infty |R_n f^+|_{\ell_1} + r_n \|x\|_\infty |R_n f^-|_{\ell_1} \\
&\leq \|x\|_\infty |P_n f|_{\ell_1} + r_n \|x\|_\infty |R_n f^+|_{\ell_1} + \|x\|_\infty |R_n f^-|_{\ell_1} \\
&= \|x\|_\infty (|P_n f|_{\ell_1} + r_n |R_n f^+|_{\ell_1} + |R_n f^-|_{\ell_1}),
\end{aligned}$$

where last inequality holds by (\heartsuit) . Thus, $f(x) \leq \|x\|_n |f|_n$ and Case 2 is completed.

Case 3. $\|R_n x^+\|_\infty \vee r_n \|R_n x^-\|_\infty = r_n |x^-(j)|$, $\|R_n x^-\|_\infty \vee r_n \|R_n x^+\|_\infty = |x^-(j)|$ for some $j \geq n+1$ and so $\|x\|_n = (1+r_n) |x^-(j)| \vee (1+r_n) \|P_n x\|_\infty = (1+r_n) \|x\|_\infty$. Case 3 can be solved using similar ideas as in Case 2.

Since the set of points considered in the above cases forms a dense set in W_{e^*} , we have shown that

$$\sup \left\{ \sum_{i=1}^{\infty} x(i) f(i) : x \in W_{e^*}, \|x\|_n \leq 1 \right\} \leq |f|_n.$$

To prove the reversed inequality, one can consider a sequence $\{x^N\}_{N>n+1}^\infty \subset W_{e^*}$ defined as follows:

$$x^N(k) = \begin{cases} (\operatorname{sgn} f_k) \frac{1}{1+r_n} & \text{for } k = 1, \dots, n, \\ \frac{-1}{1+r_n} & \text{for } f(k) < 0 \text{ and } k = n+1, \dots, N, \\ \frac{r_n}{1+r_n} & \text{for } f(k) \geq 0 \text{ and } k = n+1, \dots, N, \\ \frac{1}{1+r_n} \sum_{i=1}^n (\operatorname{sgn} f_i) e^*(i) & \text{for } k \geq N+1. \end{cases}$$

□

Proposition 2.3. $(W_{e^*}, \|\cdot\|_n)^* = (\ell_1, |\cdot|_n)$ fails the w^* -fpp.

Proof. Let $C \subset \ell_1$ be defined by

$$C = \left\{ t_0 e^* + \sum_{k=1}^{\infty} t_k e_{n+k}^* : t_i \geq 0, \sum_{i=0}^{\infty} t_i = 1 \right\}.$$

The set C is convex and weak* compact in $(W_{e^*}, \|\cdot\|_n)^* = (\ell_1, |\cdot|_n)$. Consider a mapping $T : C \rightarrow C$ given by

$$T \left(t_0 e^* + \sum_{k=1}^{\infty} t_k e_{n+k}^* \right) = \sum_{k=0}^{\infty} t_k e_{n+k+1}^*.$$

The map T is fixed point free and $|\cdot|_n$ -nonexpansive. Indeed, let

$$t = (t_0 e^*(1), \dots, t_0 e^*(n), t_1, t_2, \dots) \text{ and } s = (s_0 e^*(1), \dots, s_0 e^*(n), s_1, s_2, \dots)$$

be two elements of the set C . We consider two cases:

Case 1: $t_0 - s_0 \geq 0$.

This further implies $|R_n(t-s)^-|_{\ell_1} \geq |R_n(t-s)^+|_{\ell_1}$ and so

$$|t-s|_n = \frac{r_n}{1+r_n} |R_n(t-s)^+|_{\ell_1} + \frac{1}{1+r_n} |R_n(t-s)^-|_{\ell_1} + \frac{r_n}{1+r_n} |t_0 - s_0|.$$

Now

$$\begin{aligned} |T(t) - T(s)|_n &= \max \left\{ \frac{r_n}{1+r_n} (|t_0 - s_0| + |R_n(t-s)^+|_{\ell_1}) + \frac{1}{1+r_n} |R_n(t-s)^-|_{\ell_1}, \right. \\ &\quad \left. \frac{1}{1+r_n} (|t_0 - s_0| + |R_n(t-s)^+|_{\ell_1}) + \frac{r_n}{1+r_n} |R_n(t-s)^-|_{\ell_1} \right\} \\ &= \frac{r_n}{1+r_n} |R_n(t-s)^+|_{\ell_1} + \frac{1}{1+r_n} |R_n(t-s)^-|_{\ell_1} + \frac{r_n}{1+r_n} |t_0 - s_0| \\ &= |t-s|_n \end{aligned}$$

and so T is $|\cdot|_n$ -isometry.

Case 2: $t_0 - s_0 \leq 0$. The proof is similar with Case 1. □

Proposition 2.4. *If X is a predual of ℓ_1 with $r^*(X) \in (0, 1)$, then $\gamma^*(X) \leq \frac{2}{1+r^*(X)}$.*

Proof. Let $\epsilon \in (0, r^*(X))$ be arbitrarily chosen. There exist $e^* \in (\text{ext}(B_{\ell_1}))'$ and a subsequence $(e_{n_k}^*)_{k \geq 1}$ of the standard basis in ℓ_1 such that $1 > |e^*|_{\ell_1} > r^*(X) - \frac{\epsilon}{2}$, $(e_{n_k}^*)$ is $\sigma(\ell_1, X)$ -convergent to e^* and $|e^*|_{\ell_1} > \sum_{k=1}^{\infty} |e^*(n_k)|$.

Step 1. (Passing from X to a hyperplane in c). Let $Z = [\{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\}]$ be the norm-closed linear span of $\{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\}$, where

$$e_0^* = \frac{e^* - \sum_{k=1}^{\infty} e^*(n_k) e_{n_k}^*}{|e^*|_{\ell_1} - \sum_{k=1}^{\infty} |e^*(n_k)|}.$$

Since $\overline{\{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\}}^{w^*} = \{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\} \cup \{e^*\} \subset Z$, Lemma 1 in [2] assures that $\overline{[\{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\}]}^{w^*} = Z$. Thus $Z = (X^\perp Z)^*$. Let $y^* \in \ell_1$ be defined as

$$y^* = \left(|e^*|_{\ell_1} - \sum_{k=1}^{\infty} |e^*(n_k)|, e^*(n_1), e^*(n_2), e^*(n_3), \dots \right).$$

Since $y^* \in B_{\ell_1}$, we know that $W_{y^*}^* = \ell_1$ and $y_n^* \xrightarrow{\sigma(\ell_1, W_{y^*}^*)} y^*$, where (y_n^*) denotes the standard basis in ℓ_1 . Let ϕ be the basis to basis map of Z onto $\ell_1 = W_{y^*}^*$, that is, $\phi(a_1 e_0^* + a_2 e_{n_1}^* + a_3 e_{n_2}^* + a_4 e_{n_3}^* + \dots) = \sum_{k=1}^{\infty} a_k y_k^*$. Then $\phi(e^*) = y^*$. Indeed,

$$\begin{aligned} \phi(e^*) &= \phi \left(\left(|e^*|_{\ell_1} - \sum_{k=1}^{\infty} |e^*(n_k)| \right) e_0^* + \sum_{k=1}^{\infty} e^*(n_k) e_{n_k}^* \right) \\ &= \left(|e^*|_{\ell_1} - \sum_{k=1}^{\infty} |e^*(n_k)| \right) y_1^* + \sum_{k=1}^{\infty} e^*(n_k) y_{k+1}^* \\ &= \left(|e^*|_{\ell_1} - \sum_{k=1}^{\infty} |e^*(n_k)|, e^*(n_1), e^*(n_2), \dots \right) = y^*. \end{aligned}$$

Consequently, ϕ is a w^* -continuous homeomorphism from $\overline{\{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\}}^{w^*} = \{e_0^*, e_{n_1}^*, e_{n_2}^*, \dots\} \cup \{e^*\}$ onto $\overline{\{y_1^*, y_2^*, \dots\}}^{w^*} = \{y_1^*, y_2^*, \dots\} \cup \{y^*\}$. By Lemma 2 in [2] we see that ϕ is a w^* -continuous isometry from Z onto $\ell_1 = W_{y^*}^*$. Therefore $W_{y^*}^*$ is isometric to $X^\perp Z$. Since $\lim_n |y^* - P_n y^*|_{\ell_1} = 0$, Theorem 4.3 in [4] and

Lemma 2.1 assure that there are $n \in \mathbb{N}$ and an isomorphism $\psi : W_{y^*} \rightarrow W_{P_n y^*}$ such that $|y^* - P_n y^*|_{\ell_1} \leq \frac{\epsilon}{2}$ and $(1 - \epsilon) \|x\|_\infty \leq \|\psi(x)\|_\infty \leq (1 + \epsilon) \|x\|_\infty$ for all $x \in W_{y^*}$.

Step 2. (Renorming of a hyperplane in c). Put $r_n := |P_n y^*|_{\ell_1}$. As in Propositions 2.2 and 2.3, we consider a renorming defined for all $x \in W_{P_n y^*}$ by

$$\|x\|_n = (\|R_n x^+\|_\infty \vee r_n \|R_n x^-\|_\infty + \|R_n x^-\|_\infty \vee r_n \|R_n x^+\|_\infty) \vee (1 + r_n) \|P_n x\|_\infty,$$

a set $C \subset \ell_1$ given by

$$C = \left\{ t_0 P_n y^* + \sum_{k=1}^{\infty} t_k e_{n+k}^* : t_i \geq 0, \sum_{i=0}^{\infty} t_i = 1 \right\},$$

and a $|\cdot|_n$ -nonexpansive fixed point free mapping $T : C \rightarrow C$ defined by

$$T \left(t_0 P_n y^* + \sum_{k=1}^{\infty} t_k e_{n+k}^* \right) = \sum_{k=0}^{\infty} t_k e_{n+k+1}^*.$$

Let I from $(\ell_1, |\cdot|_n) = (W_{P_n y^*}, \|\cdot\|_n)^*$ onto $(\ell_1, |\cdot|_{\ell_1}) = (W_{P_n y^*}, \|\cdot\|_\infty)^*$ be the identity map.

Step 3. (Back to X). The mapping $\phi^{-1} \psi^* I$ is a weak* continuous isomorphism from $(\ell_1, |\cdot|_n)$ onto $(Z, |\cdot|_{\ell_1})$ and satisfies

$$(1 + r_n)(1 - \epsilon) |x|_n \leq |\phi^{-1} \psi^* I x|_{\ell_1} \leq 2(1 + \epsilon) |x|_n$$

for all $x \in (\ell_1, |\cdot|_n)$. Taking also into account the following estimate

$$\begin{aligned} r_n &= |P_n y^*|_{\ell_1} \geq |y^*|_{\ell_1} - |P_n y^* - y^*|_{\ell_1} = |e^*|_{\ell_1} - |P_n y^* - y^*|_{\ell_1} \\ &\geq r^*(X) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = r^*(X) - \epsilon > 0, \end{aligned}$$

we conclude that

$$(1 - \epsilon)(1 + r^*(X) - \epsilon)(B_{X^*} \cap Z) \subset (\phi^{-1} \psi^* I)(B_{(\ell_1, |\cdot|_n)}) \subset 2(1 + \epsilon)(B_{X^*} \cap Z).$$

Next we define the set $D \subset \ell_1 = X^*$ by

$$D = \text{conv} \left((1 - \epsilon)(1 + r^*(X) - \epsilon) B_{X^*} \cup (\phi^{-1} \psi^* I)(B_{(\ell_1, |\cdot|_n)}) \right).$$

It is easy to check that D is convex, symmetric, weak* compact, and 0 is its interior point. Therefore D is a dual unit ball of an equivalent norm $\|\cdot\|$ on X . Let $Y = (X, \|\cdot\|)$. Obviously, $D = B_{Y^*}$. Since $D \cap Z = (\phi^{-1} \psi^* I)(B_{(\ell_1, |\cdot|_n)})$, the mapping $\phi^{-1} \psi^* I$ is a weak* continuous isometry from $(\ell_1, |\cdot|_n)$ into Y^* . All the above implies that the set $(\phi^{-1} \psi^* I)(C)$ is convex, weak* compact in Y^* and fails the fpp. Finally we observe that

$$d(X, Y) \leq \frac{2(1 + \epsilon)}{(1 - \epsilon)(1 + r^*(X) - \epsilon)}.$$

$$\text{Therefore } \gamma^*(X) \leq \frac{2}{1 + r^*(X)}.$$

□

Consequently, by combining remarks made in the introduction with the thesis of Proposition 2.4, we obtain the exact value of $\gamma^*(X)$.

Theorem 2.5. *If X is a predual of ℓ_1 such that the $\sigma(\ell_1, X)$ -fpp holds, then*

$$\gamma^*(X) = \frac{2}{1 + r^*(X)}.$$

We recall that stability property for the $\sigma(\ell_1, c_0)$ -fpp was already investigated in [16]. The important point to note here is the difference between our notion of stability and the approach developed in [16]. Indeed, in that paper the author considered only the $\sigma(\ell_1, c_0)$ -topology on every renorming of ℓ_1 .

3. STABILITY IN THE RESTRICTED FRAMEWORK OF PREDUALS OF ℓ_1 : THE CASE OF c_0

In this section we deal with the stability of the w^* -fpp in the restricted framework of Lindenstrauss spaces. Namely, for a predual X of ℓ_1 that enjoys the $\sigma(\ell_1, X)$ -fpp, we are interested in the estimation of the following constant

$$\eta^*(X) = \sup \{ \eta \geq 1 : Y^* = \ell_1, d(X, Y) \leq \eta \Rightarrow Y^* \text{ has the } \sigma(\ell_1, Y)\text{-fpp} \}.$$

We restrict our attention to the case when $X = c_0$. The first step in this matter is the easy remark that $\eta^*(c_0) \leq 3$. Indeed, we know that c does not satisfy the $\sigma(\ell_1, c)$ -fpp and $d(c, c_0) = 3$ by the result in [3]. Since $\gamma^*(c_0) \leq \eta^*(c_0)$, it follows from the previous section that $2 \leq \eta^*(c_0) \leq 3$. In the sequel we prove that $\eta^*(c_0) = 3$. In order to fix some notations we recall a well-known theorem.

Theorem 3.1. (*see, e.g., [14]*). *Let $T : X \rightarrow Y$ be a linear map from the Banach space X onto the Banach space Y . Then there exists a linear map $\tilde{T} : X/\ker T \rightarrow Y$ such that*

- (1) *\tilde{T} is an onto isomorphism,*
- (2) *$T = \tilde{T}\pi$, where $\pi : X \rightarrow X/\ker T$ denotes the quotient map,*
- (3) *$\|T\| = \|\tilde{T}\|$.*

In order to prove the main theorem of this section we need some preliminary results.

First, we state the following lemma. The proof is easy and therefore we leave it to the reader.

Lemma 3.2. *Let $T : X \rightarrow Y$ be an onto bounded linear operator, where $Y \neq \{0\}$. Then*

$$\sup \{ \delta > 0 : \delta B_Y \subseteq T(B_X) \} = \|\tilde{T}^{-1}\|^{-1}.$$

Moreover, the proof of our result relies on the following theorem by Alspach.

Theorem 3.3. (*[1]*) *Let X be a quotient of c_0 . Then, for every $\epsilon > 0$, there is a subspace Y of c_0 such that $d(X, Y) < 1 + \epsilon$.*

Finally, the last result that plays a role in our argument can be viewed as an extension of Cambern's result ([3]) about the distance from c_0 to c .

Proposition 3.4. *Let X be a Banach space containing c and let $T : c_0 \rightarrow X$ be an onto linear operator with $\|T\| = 1$. Then $\|\tilde{T}^{-1}\| \geq 3$.*

Proof. By Theorem 3.3 we know that there exists a subspace Y of c_0 and an isomorphism $S : c_0/\ker(T) \rightarrow Y$ such that $\|S\| \|S^{-1}\| < 1 + \epsilon$. Let $\tilde{T} : c_0/\ker(T) \rightarrow X$ and denote by R the restriction of $S\tilde{T}^{-1}$ to c . Then R is an isomorphism from c into c_0 and by Theorem 2.1 in [11] it holds

$$3 \leq \|R\| \|R^{-1}\| \leq \|S\tilde{T}^{-1}\| \|(S\tilde{T}^{-1})^{-1}\| \leq \|\tilde{T}^{-1}\| \|S\| \|\tilde{T}\| \|S^{-1}\| \leq (1 + \epsilon) \|\tilde{T}^{-1}\|.$$

Hence we have $\|\tilde{T}^{-1}\| \geq 3$. \square

Now, we are in position to prove the main result of this section. It allows us to obtain the equality $\eta^*(c_0) = 3$.

Theorem 3.5. *Let X be a predual of ℓ_1 isomorphic to c_0 . Suppose that X^* fails the w^* -fpp. If $T : X \rightarrow c_0$ is an onto isomorphism with $\|T^{-1}\| = 1$, then $\|T\| \geq 3$.*

Proof. By Theorem 4.1 in [5] there exists a quotient X/Y of X that is isometric to a space W_α with $|\alpha|_{\ell_1} = 1$. Let us first suppose in addition that W_α contains a subspace isometric to c . Now, let us consider the onto map: $\pi T^{-1} : c_0 \rightarrow W_\alpha$, where $\pi : X \rightarrow X/Y$ is the quotient map. It is easy to check that $\pi T^{-1}(B_{c_0}) \supseteq \frac{1}{\|T\|+\varepsilon} B_{W_\alpha}$, for every $\varepsilon > 0$. Hence, by applying Lemma 3.2, we obtain

$$\left\| \left(\widetilde{\pi T^{-1}} \right)^{-1} \right\|^{-1} \geq \frac{1}{\|T\|}.$$

Moreover, since $\|\pi T^{-1}\| = 1$, Proposition 3.4 gives $\left\| \left(\widetilde{\pi T^{-1}} \right)^{-1} \right\| \geq 3$. Therefore $\|T\| \geq 3$.

Now we consider a space W_α such that c is not included in. Let us denote $(\alpha(1), \dots, \alpha(n), \sum_{i=n+1}^{+\infty} |\alpha(i)|, 0, 0, \dots)$ by α_n . Then, clearly $\lim_{n \rightarrow \infty} |\alpha - \alpha_n|_{\ell_1} = 0$. Moreover, by Proposition 2.1 in [5], the space W_{α_n} contains an isometric copy of c , for every n . By Lemma 2.1, we have that $\lim_{n \rightarrow \infty} d(W_\alpha, W_{\alpha_n}) = 1$, which completes the proof. \square

We recall that, as already said, the space c is such that ℓ_1 lacks the $\sigma(\ell_1, c)$ -fpp and $d(c_0, c) = 3$. Therefore, the equality $\eta^*(c_0) = 3$ follows directly from the previous theorem.

On the other hand, we have an example of a predual X of ℓ_1 such that the $\sigma(\ell_1, X)$ -fpp holds and $d(c_0, X) = 3$. This example is based on a space belonging to the class of hyperplanes of c already considered in Section 2. We begin with a general result on all the hyperplanes W_α with $\alpha \in S_{\ell_1}$.

Proposition 3.6. *If $\alpha \in S_{\ell_1}$, then $d(c_0, W_\alpha) = 3$.*

Proof. Let us consider the space W_α where $\alpha = (\alpha(1), \dots, \alpha(n), 0, 0, \dots) \in B_{\ell_1}$ and let $\phi : W_\alpha \rightarrow c_0$ be defined by

$$\phi(x) = \left((1+|\alpha|_{\ell_1})x(1), \dots, (1+|\alpha|_{\ell_1})x(n), x(n+1) - \sum_{i=1}^n \alpha(i)x(i), x(n+2) - \sum_{i=1}^n \alpha(i)x(i), \dots \right).$$

It is easy to check that ϕ is an onto isomorphism. Moreover, if $\alpha \in S_{\ell_1}$, then it holds

$$\|\phi\| \|\phi^{-1}\| = 3$$

and the space W_α contains an isometric copy of c (see Proposition 2.1 in [5]). Hence, Theorem 2.1 in [11] implies that $\|\varphi\| \|\varphi^{-1}\| \geq 3$ for every isomorphism φ from W_α onto c_0 . Therefore, $d(W_\alpha, c_0) = 3$ for every $\alpha \in S_{\ell_1}$ with a finite number of non null components. Now we pass to consider the general case. Let $\alpha \in S_{\ell_1}$ and $\alpha_n = (\alpha(1), \dots, \alpha(n), \sum_{i=n+1}^{+\infty} |\alpha(i)|, 0, 0, \dots)$. Then, clearly $\lim_{n \rightarrow \infty} |\alpha - \alpha_n|_{\ell_1} = 0$. By Lemma 2.1, we have that $\lim_{n \rightarrow \infty} d(W_\alpha, W_{\alpha_n}) = 1$. Therefore, we conclude that $d(c_0, W_\alpha) = 3$. \square

We are now in position to state the example announced above.

Example 3.1. *Let $\alpha = (-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \dots) \in S_{\ell_1}$. The space W_α is a predual of ℓ_1 such that $d(c_0, W_\alpha) = 3$ by Proposition 3.6. Moreover, ℓ_1 has the $\sigma(\ell_1, W_\alpha)$ -fpp by Proposition 2.2 in [5].*

By following the same arguments as in the proof of Proposition 3.6 we obtain an upper bound for the estimation of the Banach-Mazur distance between one of the considered hyperplane and c_0 . More precisely, we have

$$d(W_\alpha, c_0) \leq 1 + 2|\alpha|_{\ell_1}$$

for every $\alpha \in B_{\ell_1}$. The last inequality, combined with Theorem 2.5, gives the following characterization of stability of the $\sigma(\ell_1, W_\alpha)$ -fpp in the sense of the constant $\gamma^*(X)$.

Corollary 3.7. *Let $\alpha \in B_{\ell_1}$ be such that $\sigma(\ell_1, W_\alpha)$ -fpp holds. Then $\gamma^*(W_\alpha) > 1$ if and only if $d(W_\alpha, c_0) < 3$.*

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